Give complete solutions to the following problems. Be sure to provide all the necessary steps to support your answers.

1. Find the characteristic equation for the matrix A and determine if the solutions are real or complex, and the multiplicity of each.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$$

2. Find the eigenvalues and the eigenvectors of the matrix A, then determine both the algebraic and geometric multiplicities of each eigenvalue, and verify that

$$|\mathbf{A}| = \lambda_1 \lambda_2 \lambda_3$$

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

3. Find all eigenvectors p_1, p_2, p_3, of matrix A, then form the matrix $P = [p_1 \ p_2 \ p_3]$ and the diagonal matrix D whose diagonal entries are the eigenvalues of A in the order λ_1, λ_2, λ_3 down the diagonal, then confirm that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4. Let $\mathbf{p} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$, $v_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a. Show that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of A. [Note: A is the stochastic matrix studied in Example 3 of Section 4.9.]

b. Let $\mathbf{x}_0$ be any vector in $\mathbf{R}^3$ with nonnegative entries whose sum is 1. (In Section 4.9, $\mathbf{x}_0$ was called a probability vector.) Explain why there are constants c_1, c_2, and c_3 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Compute $\mathbf{w}^T\mathbf{x}_0$, and deduce that $c_1=1$.

c. For k = 1, 2,..., define $\mathbf{x}_k = A^k\mathbf{x}_0$, with $\mathbf{x}_0$ as in part b. Show that $\mathbf{x}_k$ approaches $\mathbf{v}_1$ as k increases.